DIFFERENTIAL GEOMETRY MID-TERM EXAM

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The marks for each question is given in brackets after the question. The total is 40.

1.(a) Show that the arc length of the catenary

$$\boldsymbol{\alpha}(t) = (t, \cosh(t)) \qquad\qquad 0 \leq t \leq b$$

is $\sinh(b)$. (4)

(b) Reparametrize the catenary by arc-length. (4)

2. Calculate $\mathbf{T}, \mathbf{N}, \mathbf{B}$ and κ, τ for the following curve. (8)

$$\boldsymbol{\alpha}(t) = (\cosh(t), \sinh(t), t)$$

3. Prove that the curvature of a plane curve $y = f(x)$ is given by (4)

$$\kappa(x) = \frac{|f''|}{(1 + f'^2)^{3/2}}$$

4. Let $\boldsymbol{\alpha}$ be a regular curve with $\kappa \neq 0$ at P. Prove that the planar curve obtained by projecting $\boldsymbol{\alpha}$ on to its osculating plane at P has the same curvature as $\boldsymbol{\alpha}$. (4)

5. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two regular curves on $[a, b]$. We say $\boldsymbol{\beta}$ is an *involute* of $\boldsymbol{\alpha}$ if, for each $t \in [a, b]$,

- $\boldsymbol{\beta}(t)$ lies on the tangent line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$
- $\boldsymbol{\alpha}'(t)$ and $\boldsymbol{\beta}'(t)$ are perpendicular for all t.

$\boldsymbol{\alpha}$ is called the *evolute* of $\boldsymbol{\beta}$. Show

(a) If $\boldsymbol{\alpha}$ is arc-length parametrized, show that

$$\boldsymbol{\beta} \text{ is the involute of } \boldsymbol{\alpha} \iff \boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + (c - s)\mathbf{T}(s)$$

for some constant c. (6)

(b) Show that the involute of a helix $(a\cos(t), a\sin(t), bt)$ is a plane curve. (6)

6. Suppose C is a simple closed plane curve with $0 < \kappa \leq c$. Prove that $length(C) \geq \frac{2\pi}{c}$. (4)